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# On a Generalization of the Law of Sines to the Tetrahedron and Simplices of Higher Dimensions

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**Abstract**

*The well known Law of Sines for triangles is carefully analyzed together with its standard proof and based on that analysis it is extended to the tetrahedron and simplices of four and more dimensions. The crucial step in the proof of the extension to the tetrahedron starts by representing each triangle in the skin (surface) of the tetrahedron as a vector in three-dimensional space whose magnitude is equal to the area of the triangle and which is normal to the plane of the triangle. The sum of these four vectors is the zero vector when the faces are properly oriented. The next step is to take the vectors and project them upon a directed line that is simultaneously perpendicular to two of the vectors. The sum of the projections must be zero, but because the directed line is orthogonal to two of them it must also be orthogonal to the sum of the vectors representing the two other faces of the tetrahedron. There are at least two simple geometrical interpretations to the main result: first, choosing two disjoint pairs of faces of the tetrahedron the edge joining the first pair of faces is orthogonal to the sum of the vectors representing the two other faces; second, the volumes of two parallelepipeds formed with trios of vector representation of faces of the tetrahedron are equal. The result is extended to simplices of n dimensions by representing the skin of the simplex by n vectors orthogonal to the hyperplanes where the elements of the skin lie. Since for n-dimensional spaces it is possible to find a vector **v** simultaneaously orthogonal to n–1 vectors, the same idea is applied and the projections of the sum of the last two vectors representing the "faces" of the skin must be orthogonal to **v.** The vector product of n – 1 vectors in n dimensional space is used to obtain **v.** A simple numerical example is given.*

***Keywords:*** *Law of Sines, tetrahedron, simplex, projection, vector product*

**Resumen**

La conocida Ley de los Senos para los triángulos es analizada cuidadosamente junto con su demostración estándar, con base en dicho análisis se le extiende al tetraedro y a simplejos de cuatro y más dimensiones. El paso crucial en la demostración de la extensión al tetraedro comienza representando cada triángulo en la piel (superficie) del tetraedro como un vector en el espacio tridimensional, cuya magnitud es igual al área del triángulo y es normal al plano en el que está el triángulo. La suma de estos cuatro vectores es el vector cero cuando

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las caras están adecuadamente orientadas. El siguiente paso es tomar los vectores y proyectarlos sobre una línea dirigida, que es simultáneamente perpendicular a dos de ellos. La suma de las proyecciones debe ser cero, pero debido a que la línea dirigida es ortogonal a dos de ellos, también debe ser ortogonal a la suma de los vectores que representan a las dos caras restantes del tetraedro. Hay por lo menos dos interpretaciones geométricas sencillas del resultado principal; primero, si se escogen dos pares disjuntos de caras del tetraedro, la arista que los une al primer par es ortogonal a la suma de los vectores que representan las caras del segundo par; segundo, los volúmenes de dos paralelepípedos formados con dos trios de vectores que representan las caras son iguales. El resultado se extiende a simplejos de cuatro y más dimensiones, representando la piel del simplejo n-dimensional por medio de $n + 1$ vectores ortogonales a los hiperplanos donde yacen los elementos de la piel. En vista de que en un espacio de $n$ dimensiones es posible encontrar un vector $\boldsymbol{v}$ simultáneamente ortogonal a $n - 1$ vectores, se aplica la misma idea y las proyecciones de la suma de los últimos dos vectores que representan las "caras" de la piel del simplejo deben ser ortogonales a $\boldsymbol{v}$. Se utiliza el producto vectorial entre $n - 1$ vectores en $n$ dimensiones para encontrar $\boldsymbol{v}$. Se muestra un ejemplo numérico.

**Descriptores:** Ley de los Senos, tetraedro, simplejo, proyección, producto vectorial.

## Introduction

The Law of Sines is one of the important theorems of Plane Geometry and Trigonometry whose importance is at a par with the Law of Cosines and is right behind the Pythagorean Theorem, which according to several authors is the most important theorem in all of mathematics (Davis and Hersh, 1980), (Wylie, 1964). In this paper we present a genealization of the Law of Sines to the tetrahedron and to analogous objects in four and more dimensions. The triangle can be seen as a particularization of a tetrahedron that has zero height. It can also be seen as the convex hull of 4 points in three dimensional space when two of the points are made to coincide and the body flattens into a two dimensional plane figure. In such a case the generalized Law of Sines for the tetrahedron reduces to the Law of Sines for the triangle. In order to generalize the Law of Sines we use the vector product of $n - 1$ vectors in $n$–dimensional space (Murray-LAsso, 2004). It is to be noted that the extension presented in this paper to the Law of Sines is an equation relating areas of the faces for the case of the tetrahedron and hyperareas of the objects in the skin of the simplex in the case of higher dimensions. Altough many extensions of the Law of Sines exist by applying the familiar Law of Sines to two-dimensional triangles that are formed or can be constructed in the tetrahedron and higher dimensional simplices, these are not treated in this paper.

## The Proof of the Law of Sines for the Triangle

In order to be able to generalize the Law of Sines to geometric objects whose dimensions

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exceeds two, it is necessary to carefully choose the proof of the original theorem and to view it in a mathematical environment as free as we can without destroying the final result.

The typical proof of the Law of Sines goes as follows (Ayres, 1954); (Gutiérrez-Ducons, 1985):

Consider the triangles shown in figure 1a and b.

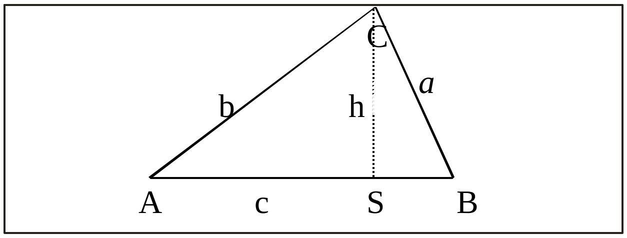


*Figure 1a*

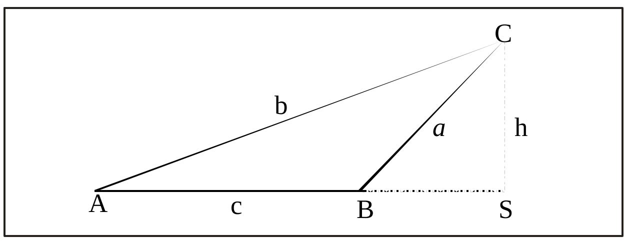


*Figure 1b*

The three vertices of the triangles are labelled A, B and C and we will use the same letters to denote the corresponding interior angles. The sides opposite to the verices are labelled with corresponding lower case letters *a*, *b* and *c*. Figure 1a represents the case in which the triangle is acuteangular while figure 1b represents the case of an obtuseangular triangle. From vertex C we draw a line perpendicular to line c — to its extension in case b) — so that we can calculate the length of line *h* in two manners:

$$h = b \sin A = a \sin B \qquad (1)$$

the last member for case b) becomes $a \sin(180 - B)$, recalling that *B* is the angle at vertex *B* *interior* to the original triangle, but since $\sin(180 - B) = \sin B$, in both cases we obtain the same expression. From the second and third member of equation (1) we obtain

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad (2)$$

We now repeat the process changing the roles of the sides of the triangle, and from vertex B we draw a perpendicular to line b and reason in a similar fashion to obtain the expression

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

which togeteher with equation (2) can be written

$$\frac{b}{\sin B} = \frac{a}{\sin A} = \frac{c}{\sin C} \qquad (3)$$

which is the *Law of Sines.*

For the purpose of generalizing the Law of Sines to more dimensions it is convenient to consider the triangle as the geometric object associated with the vector sum $a + b + c$ of three vectors (arrows) in a space with 2 or more dimensions using the *triangle law* or *polygon law* and its corresponding algebraic expressions for vector addition, that is, in the case of geometricalinterpretation the tail of a vector coincides with the arrow of the previous vector in the sum as shown in figure 2. (In figure 2 we have drawn things as though A, B and C are in the same plane, which must be the case if the space is two-dimensional. See figure 5 for the case where they are not).

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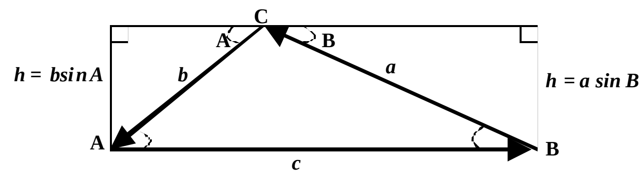


*Figure 2*

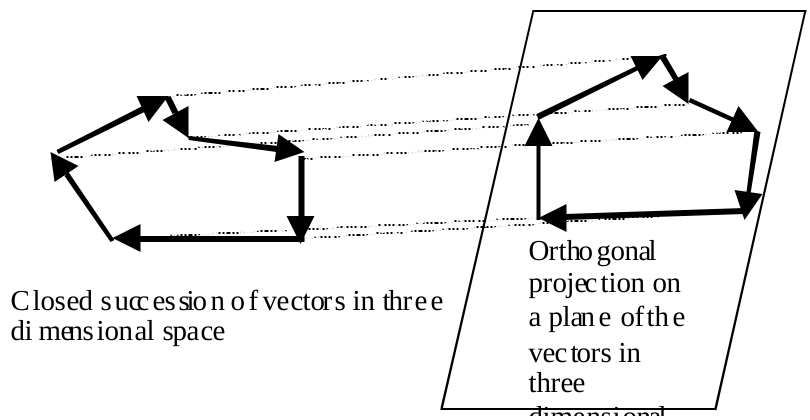


*Figure 3*

Since in the case of a triangle, the polygon is closed, the final sum will be the zero vector. Now, when we have a figure constructed from arrows succeeding each other and finally closing on itself, not only is the vector sum the zero vector, but also the vector sum of the orthogonal projections of these vectors upon *any* subspace of lower dimension. Figure 3 illustrates the idea. In the proof of the Sine Law the space for projecting is a line orthogonal to one of the sides, say the horizontal side. In this case the horizontal line has zero orthogonal projection upon the line orthogonal to it and only two sides have a non zero projection. Since the two projections must add to zero, the magnitudes of the proyections must be equal.

The projections of vectors ***a*** and ***b*** upon a line orthogonal to ***c*** do not have to be thought of as passing through point *C,* the important point is that they have the same magnitude; their values are $h = |\,\boldsymbol{b}\,|\, \sin A = |\,\boldsymbol{a}\,|\, \sin B,$ (the angles marked *A* and *B* are equal to the interior angles at the vertices A and B because they are alternate interior angles between parallel lines) from which the Law of Sines follows by repeating the argument for lines orthogonal to either side ***b*** or side ***a.*** It is this interpretation that will allow us to see a generalization of the Law of Sines to higher dimensional objects.

## The Law of Sines for the Tetrahedron

To generalize the Law of Sines to the tetrahedron we represent each of its faces with a vector whose magnitude is the absolute value of the area of the face and whose direction is ortogonal to the plane of the face (Spiegel, 1998). To decide on the direction of the vector we orient the faces by defining a sense in

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which the rim of the face is traversed (this can be done by ordering the vertices in a particular way) and associating the arrows of the vectors with the "right hand screw rule," which says that the arrow points in the direction in which a right hand screw would advance when it is turned in the sense in which the rim of the surface is traversed (Spiegel, 1998). This is shown in figure 4

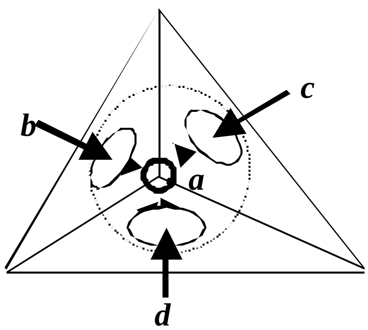


*Figure 4*

In figure 4 we are observing a tetrahedron as seen from above. The orientations of the faces are so chosen that all arrows enter the tetrahedron. The arrow corresponding to the horizontal plane is drawn as a circle with a dot in the middle to represent an arrow pointing directly towards the observer. The dotted directed circle is the direction of traversal of the horizontal face. The vector sum of the four arrows representing the faces in three dimensional space have a zero sum, therefore, when arranged in space according to the polygon law, they form a four-edge three-dimensional polygon that closes upon itself (Spiegel, 1998). As mentioned above, the projection of this closed arrow polygon upon any subspace must be the zero vector in the said subspace. To look for a sine law we choose a subspace for projecting consisting of a line such that only two of the vectors representing the faces of the tetrahedron have non zero projections upon the line.

Since we are in a three dimensional space, if we form the vector product of two of the vectors representing the faces of the tetrahedron, the resulting vector will be orthogonal to both vector factors and both will have zero projections along the line. The projections on the line of the other two face-representing vectors (which can be obtained through a dot product) must be equal in magnitude. Let us choose the face-representing vectors ***a*** and ***b*** as the vectors that will have zero projections, then equating the magnitudes of the projections of the other two vectors we obtain the following expression

$$|\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}| = |\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{d}|\,* \qquad (4)$$

where we have cancelled a factor $|\,\boldsymbol{a}\times\boldsymbol{b}\,|$ dividing both members of equation (4). By choosing other pairs of vectors for the cross product and repeating the argument we obtain (the meaning of the asterisk´s is explained below)

$$|\boldsymbol{a}\times\boldsymbol{c}\cdot\boldsymbol{b}| = |\boldsymbol{a}\times\boldsymbol{c}\cdot\boldsymbol{d}|\,* \qquad (5)$$

$$|\boldsymbol{a}\times\boldsymbol{d}\cdot\boldsymbol{c}| = |\boldsymbol{a}\times\boldsymbol{d}\cdot\boldsymbol{b}| \qquad (6)$$

$$|\boldsymbol{c}\times\boldsymbol{b}\cdot\boldsymbol{a}| = |\boldsymbol{c}\times\boldsymbol{b}\cdot\boldsymbol{d}|\,* \qquad (7)$$

$$|\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{d}| = |\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a}| \qquad (8)$$

and several others. The equations are not all independent due to the properties of the triple product. Using the brackett notation (Hsu, 1986) equations (7) and (8) can be written

$$|\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}| = |\boldsymbol{c},\boldsymbol{b},\boldsymbol{d}|$$

$$|\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}| = |\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}|$$

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Taking into account that a cyclic permutation of the letters in the bracketts does not change its value and that an ordinary permutation changes the sign (Hsu, 1986) the last equation can be written

$$-[\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}] = -[\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{a}]$$

Cancelling both negative signs in this last equation reveals that equation (8) is really the same equation as (7). Only the equations marked with an asterisk are independent in the set of equations (4) to (8). The three equations with asterisk (or other equivalent ones) are a vector form of the Law of Sines for the tetrahedron. Notice that it is an equation relating areas of faces, not lengths of lines.

## Geometric Interpretations

Equation (4) can be given the following interpretation:

The cross product $\boldsymbol{a} \times \boldsymbol{b}$ produces a vector that is orthogonal to both **a** and **b.** Being orthogonal to vector $\boldsymbol{a}$ implies it lies on a plane parallel to (that is with the same orientation) as face A of the tetrahedron. Being orthogonal to vector $\boldsymbol{b}$ implies it lies in a plane with the same orientation as face B of the tetrahedron. Imposing both conditions simultaneously means it lies in a line whose orientation is the same as the intersection of faces A and B, that is, the edge that joins faces A and B. The vector can be normalized to unit length by dividing it by the scalar $|\boldsymbol{a} \times \boldsymbol{b}|$ (although that will not be necessary because this factor can be cancelled with the same factor appearing on the right member.) The dot product (of the normalized vector) with vector $\boldsymbol{c}$ gives simply the projection of $\boldsymbol{c}$ upon the edge mentioned. This takes care of the left side of equation (4). Similarily the right hand side represents the projection of the vector $\boldsymbol{d}$ upon the same edge (as summing we still have not taken out the normalizing factor $|\boldsymbol{a} \times \boldsymbol{b}|$). After cancelling the normalizing factor on both sides of equation (4) what we have is the magnitudes of both projections multiplied by the factor $|\boldsymbol{a} \times \boldsymbol{b}|$. The interpretation can be applied to to other pairs of vectors representing the areas of the faces of the tetrahedrons. We note that to obtain any of the formulas we choose the vectors associated with a a pair of faces; their cross product determines a vector in the direction of the edge between the faces. The magnitudes of the dot products of the two vectors associated with the remaining two faces are then equal to each other.

Since the algebraic sum of the projections of the four vectors representing the faces of a tetrahedron upon any directed line is zero and when the line chosen is orthogonal to two of them, say $\boldsymbol{a}$ and $\boldsymbol{b}$, then the sum of the projections of vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ gives the zero vector. This means that the following equation holds

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{0} \tag{9}$$

in other words, the cross product of any two of the vectors representing two faces are orthogonal to the sum of the vectors representing the two remaining faces.

A second interpretation can be given in terms of volumes, since a triple product can be associated with the volume of a parallelepiped whose sides meeting at a vertex are the vectors in the triple product. The Law of Sines can then be interpreted as the equality between the volumes of the parallelepipeds defined by vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}$. Several additional equalities between volumes can be obtained by permutting the vectors. Recall that the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are vectors

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orthogonal to the faces of the tetrahedron of magnitudes equal to their areas.

The reader may wonder whether we can obtain for the tetrahedron expressions similar to the equations appearing in the Law of Sines for triangles. We certainly can, since for any closed four sided polygon of three-dimensional vectors closing upon themselves (that is, such that the sum of the vectors is the zero vector) the algebraic sum of the projections of the vectors with respect to a directed line or thogonal to two of them (which for non zero vectors can always be obtained through the cross product) is zero, we have a situation such as that depicted in figure 5.

The projections of ***a*** and ***b*** upon ***z*** are given by

$$Proj_z\, \mathbf{a} = \mathbf{a} \cdot \mathbf{z} / |\mathbf{z}| = \mathbf{n} |\mathbf{a}| \cos\theta = \mathbf{n} |\mathbf{a}| \sin\beta$$

$$Proj_z\, \mathbf{b} = \mathbf{b} \cdot \mathbf{z} / |\mathbf{z}| = \mathbf{n} |\mathbf{b}| \cos\varphi = \mathbf{n} |\mathbf{b}| \sin\alpha$$

Where ***n*** is a unit vector in the direction of ***z***. Since both projections are equal in magnitude we have

$$|\mathbf{b}|\,|\sin\alpha| = |\mathbf{a}|\,|\sin\beta|$$

from which a part of the Law of Sines is obtained in the form

$$\frac{|\mathbf{a}|}{|\sin\alpha|} = \frac{|\mathbf{b}|}{|\sin\beta|}$$

with the geometric interpretation that the angles are between vectors representing two faces of the tetrahedron and perpendicular lines in the same plane as a line which is simultaneously orthogonal to the two other faces and the vector such as ***a*** representing a face. By applying the same argument to additional pairs of vectors representing faces we can deduce a series of expressions of the form

$$\frac{a}{|\sin\alpha|} = \frac{b}{|\sin\beta|};\ \frac{c}{|\sin\gamma|} = \frac{d}{|\sin\delta|};\ \frac{e}{|\sin\varepsilon|} = \frac{f}{|\sin\kappa|},\ldots$$

where the letters *a, b, c, d, e, f* represent magnitudes of vectors and the angles have similar geometric interpretations. It is necessary to introduce different angles for different pairs of faces

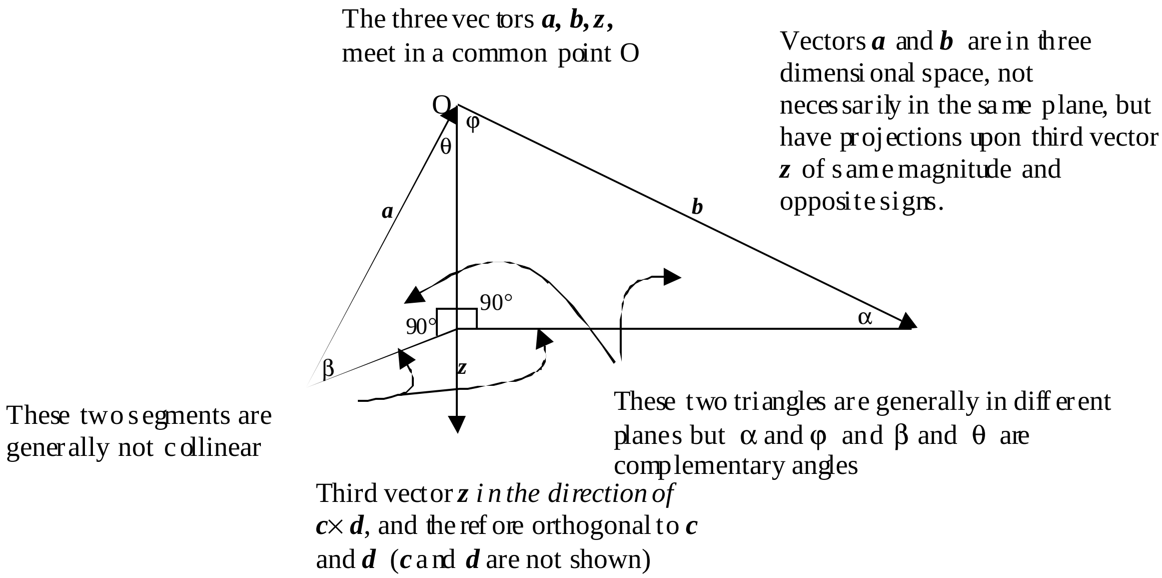


*Figure 5*

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because their cross product determines different vectors playing the role of $\mathbf{z}$ in figure 5.

## Numerical Illustrative Example

Consider a tetrahedron such as the one shown in figure 6, in which the coordinates with respect to Cartesian axes x, y, z are given for the vertices A, B, C, D.

In figure 6 the vertices of the tetrahedron are labelled A, B, C, D and close to them are trios of number in parentheses with their coordinates. The vectors representing the faces of the tetrahedron are labelled *a, b, c, d* while the faces themselves are labelled A', B', C', D'. Each of the edges has been given a sense. The sense of traversal of each of the faces is such that the vectors representing them are all directed leaving the tetrahedron. The vectors representing the edges are as follows:

*Edge DC : (0.5, 1, 0) – (0, 0, 0) = (0.5, 1, 0)*
*Edge DB : (1, 0, 0) – (0, 0, 0) = (1, 0, 0)*
*Edge AB : (1, 0, 0) – (0.2, 0,1) = (0.8, 0, –1)*
*Edge AC : (0.5, 1, 0) – (0.2, 0,1) = (0.3, 1, –1)*
*Edge DA : (0.2, 0, 1) – (0, 0, 0) = (0.2, 0, 1)*

The vector representation *v* of a face by a vector normal to it and whose magnitude is given by the area of the face, in the case of triangles is given by

$$\mathbf{v} = \tfrac{1}{2}\,(\mathbf{v_1} \times \mathbf{v_2})$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are two adjacent edges of the triangles and, given a sense of traversal of the rim of the triangle, the order of the factors in the product is chosen so that when the first vector is rotated through an angle less than 180 degrees, so as to make it coincide in direction and sense with the second vector, the turning is in the sense of traversal of the rim of the area represented. Using these concepts we find:

$\mathbf{a} = \tfrac{1}{2}\,(BC \times DC) = (0, 0, -0.5)$
$\mathbf{b} = \tfrac{1}{2}\,(DA \times DC) = (-0.5, 0.25, 0.1)$
$\mathbf{c} = \tfrac{1}{2}\,(DB \times DA) = (0, -0.5, 0)$
$\mathbf{d} = \tfrac{1}{2}\,(AB \times AC) = (0.5, 0.25, 0.4)$

Notice that the sum of the four vectors representing the faces is the zero vector.

We now proceed to check whether equation (4) is satisfied.

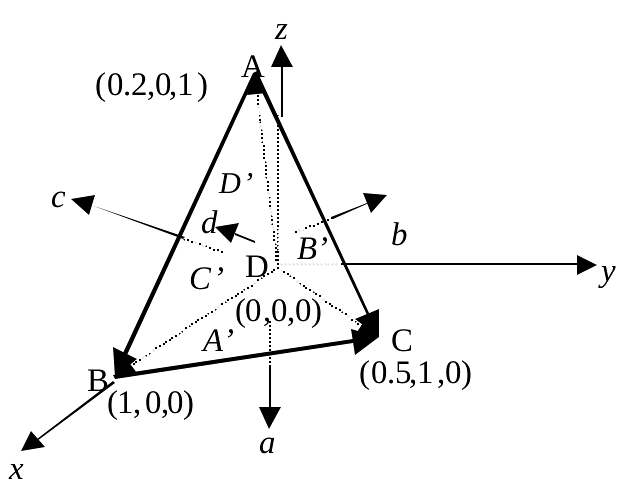


*Figure 6*

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$$\mathbf{a} \times \mathbf{b} = (0.125, 0.25, 0)$$

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = -0.125 \qquad \mathbf{a} \times \mathbf{b} \cdot \mathbf{d} = 0.125$$

Since the two magnitudes are equal, equation (4) is satisfied. The orthognality property can be checked in a second way. The vector $DC$ between faces **a** and **b** is

$$(0.5, 1, 0)$$

This vector must be orthogonal to the sum **c**+**d.**

$$\mathbf{c} + \mathbf{d} = (0, -0.5, 0) + (0.5, 0.25, 0.4) = (0.5, -0.25, 0.4)$$

The orthogonality between the two vectors can be checked via the dot product

$$(0.5, 1, 0) \cdot (0.5, -0.25, 0.4) = 0.25 - 0.25 = 0$$

Since the dot product is zero, the two vectors are orthogonal. In a similar fashion we obtain

$$|\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}| = |\mathbf{c} \times \mathbf{b} \cdot \mathbf{d}| = 0.125$$

therefore equation (7) is satisfied. By choosing different pairs of vectors the Law can be tested for the other cases. We leave the verifications to the reader.

## Extension to More Dimensions

The extension of the Law of Sines to simplices of more dimensions is straightforward. A simplex in four dimensions, for instance, is a geometric object that has a three dimensional skin consisting of 5 three-dimensional tetrahedrons (which is the number of possible combinations of 5 points taken 4 at a time), in analogy to a tetrahedron in three dimensions that has a two-dimensional skin consisting of 4 ( number of combinations of 4 points taken 3 at a time) two-dimensional triangles. Each one of the tetrahedrons of the four-dimensional simplex is a piece of a 3-dimensional hyperplane and can be represented in four-dimensional space through a four-dimensional vector whose direction is orthogonal to the hyperplane and whose magnitude is equal to the volume of the corresponding tetrahedron. Now in a four-dimensional space we can find a vector which is simultaneously orthogonal to three four-dimensional vectors (a generalization to one more dimension of the fact that in a *two* dimensional space we can find a vector orthogonal to *one* vector; in a *three* dimensional space we can find a vector that is simultaneously orthogonal to *two* vectors.) One way to find them is solving a set of homogeneous linear equations with a zero determinant. A second way is through the cross product of *three* vectors in 4-dimensions. A convenient definition for this product in terms of the cartesian components of the four-dimensional vectors **a**, **b**, and **c** is

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_2 & c_4 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \end{vmatrix}$$

where **i**, **j**, **k**, **l** are unit vectors in the direction of an orthogonal right-handed system of axes. The result of the cross product of the three vectors is a four dimensional vector, that it is orthogonal to all three vectors can be easily seen by performing the dot product with each one of them in succession. The

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mixed product $(\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{d}$ is a scalar that can be written

$$(\boldsymbol{i}\, p_1 + \boldsymbol{j}\, p_2 + \boldsymbol{k}\, p_3 + \boldsymbol{l}\, p_4) \cdot (\boldsymbol{i}\, d_1 + \boldsymbol{j}\, d_2 + \boldsymbol{k}\, d_3 + \boldsymbol{l}\, d_4) = p_1 d_1 + p_2 d_2 + p_3 d_3 + p_4 d_4$$

where $p_1, p_2, p_3, p_4$ are the components of the product $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$ and $d_1, d_2, d_3, d_4$ are the components of $\boldsymbol{d}$. This last scalar can be written as the following determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

If $\boldsymbol{d}$ is equal to any of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ the determinant will have two identical rows, and therefore according to a well known theorem of determinants (Thomas, 1960) the value of the determinant will be zero, establishing the orthogonality of the three-vector cross product with each of its factors.

Once we know how to find a vector orthogonal to three vectors in four dimensional space, given the five four-dimensional vectors representing the tetrahedra that forms the skin of the four-dimensional simplex, vectors that when we properly orient the tetrahedra add to the zero vector, we find a vector orthogonal to three of them and with it and the two remaining vectors representing two of the tetrahedra we form two triangles in four dimensions for which figure 5 applies and hence we can extend the Law of Sines to this case. The expression of the Law can be written

$$(\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}) \cdot (\boldsymbol{d} + \boldsymbol{e}) = 0$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ are four-dimensional vectors representing the tetrahedra of the skin of the four-dimensional simplex. Many similar expressions can be obtained by interchanging the vectors.

Generalizing to $n$ dimensions, for a simplex in $n$ dimensions whose skin is formed by $n + 1$ simplices of $n - 1$ dimensions represented by $n + 1$ vectors orthogonal to their hyperplanes, the expressions associated with the Law of Sines take the form

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2 \times \ldots \times \boldsymbol{v}_{n-1}) \cdot (\boldsymbol{v}_n + \boldsymbol{v}_{n+1}) = 0$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n+1}$ are the vector representations of the $(n - 1)$-dimensional simplices conforming the skin of the $n$-dimensional simplex. Additional expressions may be obtained by permuting the indices of the vectors.

The last equation is also valid for a two-dimensional simplex, that is for a triangle, if the vector product is properly interpreted. The vector expression for the triangle reads

$$\boldsymbol{a} \times \cdot (\boldsymbol{b} + \boldsymbol{c}) = 0$$

In this case the vector product has only one factor and its calculation can be done using the determinantal expression

$$\begin{vmatrix} a_1 & a_2 \\ \mathbf{i} & \mathbf{j} \end{vmatrix} = -\mathbf{i} a_2 + \mathbf{j} a_1$$

which is orthogonal to the vector $\boldsymbol{i}\, a_1 + \boldsymbol{j}\, a_2$ which plays the role of the horizontal side of the triangle in the original proof of the Sine Law. It is for reasons such as this that it is more convenient to consider the vector product in $n$ dimensions as a function of $n - 1$ ordered variables than as a binary operation and therefore the notation $\times(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{c})$ is to be preferred to $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$.

The sides of the triangle represented by orthogonal vectors is shown in figure 7.

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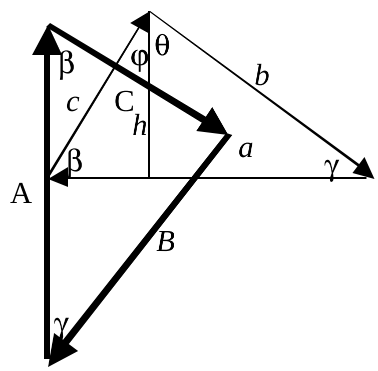


*Figure 7*

In figure 7 the original triangle is shown in heavy lines, its sides are labelled with upper case letters. The vectors orthogonal to the sides of the original triangle, rotated positively through 90 and of magnitudes equal to the lengths of the original sides are labelled with the corresponding lower case letters. The two triangles are congruent, one is simply a rotation of the other, hence $A = a$, $B = b$, $C = c$ and the angles labelled with the same letters are equal. Notice also that both triangles are traversed in a clockwise direction when the traversal coincides with the senses of the arrows. Since the arrows add to the zero vector the conditions stated for the proof of the Law of Sines hold. The horizontal side of the triangle in thin lines *a* is chosen as the side that will have a zero projection upon a vertical line (orthogonal to the horizontal side *a*). The projections of the sides *b* and *c* upon the vertical line *h* are

$$b \cos\theta = b \sin\gamma = c \cos\varphi = c \sin\beta$$

from which

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} = \frac{C}{\sin\gamma} = \frac{B}{\sin\beta}$$

and the rest of the proof follows by now taking a second side as the one having zero projection upon a line orthogonal to the side.

## Conclusions

We have extended the familiar Law of Sines of the triangle to the tetrahedron and to simplices of four and more dimensions. The law relates the areas and hyperareas of the elements of the skin of the body. Hence for the tetrahedron it relates areas of triangular faces, for the 4-dimensional simplex it relates volumes of tetrahedra forming the skin of the simplex, etc. To do it conveniently we established a vector expression that is equivalent to the Law of Sines. The proof was accomplished by taking a slightly different view of the ingredients in the standard proof of the Law of Sines for the triangle. The vector product of *n–1* vectors in *n*-dimensional space came in very handy for obtaining a vector that is simultaneously orthogonal to all the vector factors which is an essential part of the proof of the Law of Sines.

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## Semblanza del autor

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